Math 254A Lecture 22 Notes

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1 Basics of Lattice Models

1.1 Lattices

In the derivation of van der Waal's equation, we used a discretization and let $\varepsilon \to 0$. Now we will begin looking at lattice models, where there is a fixed lattice; this precludes any notion of particles getting too close to each other. Consider boxes $B_n \subseteq \mathbb{Z}^d$, which are cuboids with all sides $\to \infty$. At each site $i \in B_n$, the set of possible "local states" is a finite set A (the **alphabet**). So the microscopic states of the system are elements $\omega \in \Omega_n = A^{B_n}$.

Example 1.1 (Lattice gases)**.** If $A = \{0, 1\}$, we can interpret "$\omega_i = 1$" as "there is a particle at i" and interpret "$\omega_i = 0$" as 'there is no particle at i."

Example 1.2 (Magnetizable solid)**.** If $A = \{-1, 1\}$, we can interpret ω_i as the "direction" of a magnetic spin located at i inside a magnetizable solid. (More realistic magnet models allow A to be a sphere in $\mathbb{R}^3$.)

Example 1.3. More generally, we could have $A = \{0, a_1, a_2, \dots, a_k\}$. Here, 0 represents the absence of a particle, and the a_i represent possible internal states of particles.

1.2 Interactions

The total energy of $\omega \in \Omega_n$ will be given by an *interaction*.

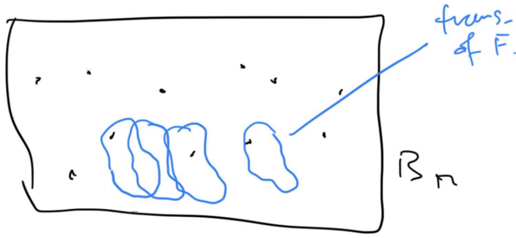
Definition 1.1. An **interaction** is a family $(\varphi_F : F \subseteq \mathbb{Z}^d,\ F \text{ finite})$, where

1. $\varphi_F : A^F \to \mathbb{R}$
2. translation invariant:

$$\varphi_F(\underbrace{(a_v)_{v \in F}}_{\in A^F}) = \varphi_{F+u}(\underbrace{(a_{v+u})_{v \in F}}_{\in A^{u+F}}).$$

Then for $\omega \in \Omega_n$, its **total (potential) energy** is

$$\Phi_{B_n}(\omega) = \sum_{F \subseteq B_n} \varphi_F(\omega_F).$$



Example 1.4. Most simply, a **pair interaction** has $\varphi_F = 0$ unless $|F| = 1$ or $|F| = 2$.

For example, as in our study of van der Waal's equation, we could take $A = \{0, 1\}$ and

$$\varphi_F(\omega) = \begin{cases} -\varphi^r(i-j)\omega_i\omega_j & F = \{i,j\} \\ 0 & \text{otherwise.} \end{cases}$$

1.3 Interaction decay

We want to understand asymptotic behavior as $B_n \to \mathbb{Z}^d$. This is known as the **thermodynamic limit**. To get a meaningful limit, we need enough decay in interaction strength with distance. Possible additional assumptions are:

1. **Finite range**: There exists some $R < \infty$ such that $\varphi_F = 0$ if $\operatorname{diam} F \geq R$.
2. A bit more general: φ is in the **big space** of interactions if $\sum_{F \ni 0} \frac{\|\varphi_F\|_\infty}{|F|} < \infty$. This guarantees "finite energy per particle."
3. φ is in the **small space** of interactions if $\sum_{F \ni 0} \|\varphi_F\|_\infty < \infty$. This is more restrictive than the big space.

Note that the big space and small space of interactions are Banach spaces, and these quantities are norms. We will tend to prove results assuming finite range, with the understanding that a bit more careful reasoning will work for the more general assumptions.

So now we need to look at sets of the form

$$\Omega_{B_n}(\varphi, I) = \left\{ \omega \in A^{B_n} : \frac{\Phi_{B_n}(\omega)}{|B_n|} \in I \right\},$$

for $I \subseteq \mathbb{R}$. Here, we are keeping track of energy per unit volume.

1.4 Observables

Next, we need a notion of macroscopic observables. We will study these as "averages over B_n."

Definition 1.2. An **observable** is a function $\psi : A^W \to \mathbb{R}$ with $W \subseteq \mathbb{Z}^d$, and

$$\Psi_{B_n}(\omega) = \sum_{i+W \subseteq B_n} \psi(\omega_{i+W}).$$

Example 1.5. If $A = \{0,1\}$, $W = \{0\}$, and $\psi(a) = a$, then

$$\Psi_{B_n}(\omega) = \sum_{i \in B_n} \omega_i = \#\text{ particles in } B_n.$$

1.5 Entropy

We want to study the growth of the cardinality of

$$\Omega_{B_n}(\varphi, I; \psi, J) = \left\{ \omega \in A^{B_n} : \frac{\Phi_{B_n}(\omega)}{|B_n|} \in I, \frac{\Psi_{B_n}(\omega)}{|B_n|} \in J \right\},$$

where I is an open interval $\subseteq \mathbb{R}$, and J is an open convex subset of $\mathbb{R}^n$.

Theorem 1.1. *Let $s_n(I,J) = \log |\Omega_{B_n}(\varphi, I; \psi, J)|$. Then there exists a concave and upper semicontinuous function $s : \mathbb{R} \times \mathbb{R}^r \to [-\infty, \infty)$ such that*

$$s_n(I,J) = |B_n| \cdot \sup_{(x,y) \in I \times J} s(x,y) + o(|B_n|).$$

We will prove this assuming φ has finite range. In this case, we will simplify our work by studying

$$\frac{\sum_{i+F \subseteq B_n} \varphi_F(\omega_{i+F})}{|B_n|} \in I_F$$

for every $\operatorname{diam}(F) < R$, rather than

$$\sum_{\substack{\text{equiv. classes of} \\ \operatorname{diam} F < R \text{ up} \\ \text{to translation}}} \frac{\sum_{i+F \subseteq B_n} \varphi_F(\omega_{i+F})}{|B_n|} \in I.$$

This lets us write

$$\Omega_{B_n}(\psi, J) = \left\{ \frac{\Psi_n(\omega)}{|B_n|} \in J \right\}$$

for a single observable $\psi : A^W \to \mathbb{R}^{r'}$ with r' bigger than r. Let's restate the theorem:

Theorem 1.2. *In the setting above,*

$$s_n(\psi, J) = |B_n| \cdot \sup_{x \in J} s(x) + o(|B_n|),$$

where $s : \mathbb{R}^r \to [-\infty, \infty)$ is concave and upper semicontinuous.

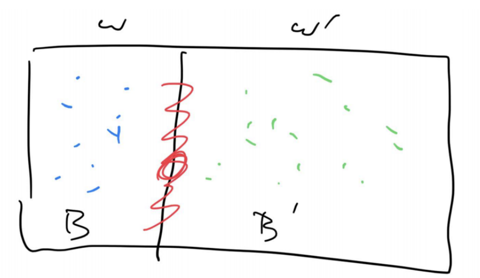
We would like to show that $S_n(\psi, J)$ is superadditive. In fact, previously, we had

$$s_n(\psi, J) + s_m(\psi, J') \leq s_{n+m}\left(\psi, \frac{n}{n+m}J + \frac{m}{n+m}J'\right)$$

for non-interacting systems. First, we need a version for cuboids, something like

$$s_B(\psi, J) + s_{B'}(\psi, J') \leq s_{B \cup B'}\left(\psi, \frac{|B|}{|B| + |B'|}J + \frac{|B'|}{|B| + |B'|}J'\right),$$

when



But

$$\Psi_{B \cup B'}(\omega, \omega') = \Psi_B(\omega) + \Psi_{B'}(\omega') + \text{boundary terms}.$$

We will have to take care of these boundary terms of make this argument work in this case.